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"EXACT" TWO-SIDED CONFIDENCE INTERVALS ON NONNEGATIVE

LINEAR COMBINATIONS OF VARIANCES

by

Franklin A. Graybill and Chih-Ming Wang

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A method is presented for obtaining two-sided confidence intervals on a nonnegative linear combination of two variances and the confidence coefficient is "exact" (equal to a specified value $1-\alpha$).

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"EXACT" TWO-SIDED CONFIDENCE INTERVALS ON NONNEGATIVE LINEAR COMBINATIONS OF VARIANCES

by

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1. Introduction

In a paper to soon be published in the Journal of the American Statistical Association a method is presented for determining approximate confidence intervals on nonnegative linear combinations of variances. In section 2 is a summary of that paper. In this paper we present a method for obtaining "exact" confidence intervals for the problem.

2. Derivation of the Confidence Interval

Let $n_i S_i^2/\theta_i$ for $i = 1, 2, \ldots, K$ be independently distributed as chi-square random variables with n_i degrees of freedom respectively.

There are no exact (the word "exact" means exact specified confidence coefficient) confidence intervals available for nonnegative linear combinations of the θ_i. Smith (1936) defined an estimate of a linear function of variances to be a linear function of independent mean squares and proposed approximating the distribution of such an estimate by a chi-square distribution whose degrees of freedom are determined by equating the variance of the estimate to the variance of the approximating (chi-square) random variable. From this distribution one can obtain approximate confidence intervals on linear functions of variances. Satterthwaite (1941, 1946) studied this approximation and it has become known as the

Satterthwaite procedure. Welch (1956) exhibited a series approximation, analogous to the Cornish-Fisher expansion, for the general problem of finding confidence limits for linear combinations of several variances. Huitson (1955) also gave a method for setting confidence intervals on linear combinations of variances. He arrives at some of the methods presented by Welch, although the details of their derivations differ considerably. Huitson includes a special set of tables which must be used to obtain the confidence intervals. Fleiss (1971) discusses the Satterthwaite and Welch methods for setting confidence limits on $\sigma_A^2 + \sigma_e^2$ for the two factor cross component-of-variance model and arrives at the conclusion that Welch's method is adequate (and better than Satterthwaite's method). Fleiss only evaluates the cases where $n_2 = 2n_1$. However, when n_2 is large relative to n_1, Welch's procedure may not be very good. This is demonstrated in Table 1. This table was obtained by numerical integration and the entries are the ranges over which the confidence coefficients vary as the unknown parameter $\rho = c\theta_1/(c\theta_1 + \theta_2)$ varies from 0 to 1. The nominal confidence coefficient is $1 - \alpha = .95$. In component-of-variance models in applied problems it is often the case that n_2 is much larger than n_1 so the conclusions given by Fleiss may not apply in those cases.

Burdick and Sielken (1978) propose a method that can be used for exact confidence intervals for this problem, but the expected lengths of their intervals are extremely bad. They are sometimes more than 800% larger than the expected widths given by Satterthwaite, so their method cannot be recommended for the problem discussed in this paper.

The purpose of this section is to describe and discuss the method, called the Modified Large Sample (MLS) method, for obtaining confidence intervals on $\theta = \sum_{i=1}^{K} c_i\theta_i$ with nonnegative constants c_i. The procedure proposed here is compared to those of Satterthwaite and Welch.

To illustrate the method we first discuss it for a linear combination of two variances, i.e. for $\theta = c\theta_1 + \theta_2$. The UMVU estimator $\hat{\theta}$ of θ is $cS_1^2 + S_2^2$, and $\operatorname{var}[\hat{\theta}] = c\ (2\theta_1^2/n_1) + 2\theta_2^2/n_2$. Thus $Z = (\hat{\theta}-\theta)/\sqrt{\operatorname{var}[\hat{\theta}]}$ has a limiting normal distribution with mean zero and variance one as $\min(n_1, n_2) \to \infty$. Using these results an approximate $1 - \alpha$ confidence interval on θ is given by

$$cS_1^2 + S_2^2 - N_\alpha\sqrt{c^2(2\theta_1^2/n_1) + 2\theta_2^2/n_2} \le \theta \le cS_1^2 + S_2^2 + N_\alpha\sqrt{c^2(2\theta_1^2/n_1) + 2\theta_2^2/n_2}$$

where N_α is the upper α probability point of a standard normal p.d.f.

To utilize these limits, we replace θ_1^2 and θ_2^2 by S_1^4 and S_2^4 respectively. We then modify the confidence limits so they might be more exact for small or moderate sample sizes by replacing the constants $-N_\alpha$, N_α, $2/n_1$, $2/n_2$ by general constants and obtain the following for the approximate $1 - \alpha$ confidence interval on θ

$$cS_1^2 + S_2^2 - \sqrt{L_1^2c^2S_1^4 + L_2^2S_2^4} \le \theta \le cS_1^2 + S_2^2 + \sqrt{H_1^2c^2S_1^4 + H_2^2S_2^4}. \quad (2.1)$$

We now determine L_1, L_2, H_1, H_2 by forcing the confidence interval to have an exact confidence coefficient $1 - \alpha$ when $\theta_1 = 0$ and when $\theta_2 = 0$. When $\theta_i = 0$ it follows that $S_i^2 = 0$ with probability one so we obtain $L_i = 1 - 1/F_{\alpha_{i1}:\, n_i,\, \infty}$, $H_i = 1/F_{\alpha_{i2}:\, n_i,\, \infty} - 1$ for $i = 1$, 2 where $F_{\gamma:\, m,\, n}$ is the upper γ probability point of Snedecor's F distribution with m degrees of freedom in the numerator and n degrees of freedom in the denominator. Also $\alpha_{i1} > 0$, $\alpha_{i2} > 0$, $\alpha_{i1} + \alpha_{i2} = 1$. The resulting confidence interval on $c\theta_1 + \theta_2$, called the Modified Large Sample (MLS) confidence interval, is in (2.1).

The α_{ij} can be chosen so that when $\theta_i = 0$ for either $i = 1$ or $i = 2$ the resulting confidence interval satisfies one of the following three conditions. (1) "Equal tails" confidence intervals (we denote this method by MLS1 and in this case $\alpha_{i1} = \alpha/2$ and $\alpha_{i2} = 1 - \alpha/2$ for $i = 1, 2$); (2) "Shortest unbiased" confidence intervals (we denote this method by MLS2 and for values of L_i, H_i see John (1973)); (3) "Shortest" confidence intervals (we denote this method by MLS3 and for values L_i, H_i see Tate and Klett (1959)).

Note that the confidence interval in (2.1) is also exact when $n_1 \to \infty$ and n_2 is fixed, or when $n_2 \to \infty$ and n_1 is fixed.

To generalize the MLS procedure to nonnegative linear combinations of K variances, we proceed as follows:

a) $U_i = n_i S_i^2/\theta_i$ are independent chi-square random variables with n_i degrees of freedom for $i = 1, 2, \ldots, K$.

b) define θ by $\theta = \sum_{i=1}^{K} c_i\theta_i$ where $c_i \geq 0$, $c_K = 1$;

c) an approximate $1 - \alpha$ confidence interval on θ is

$$\Sigma c_i S_i^2 - \sqrt{\Sigma L_i^2 c_i^2 S_i^4} \leq \theta \leq \Sigma c_i S_i^2 + \sqrt{\Sigma H_i^2 c_i^2 S_i^4} \qquad (2.2)$$

where $L_i = 1 - 1/F_{\alpha_{i1}:\, n_i,\, \infty}$; $H_i = 1/F_{\alpha_{i2}:\, n_i,\, \infty} - 1$ where $\alpha_{i1} > 0$, $\alpha_{i2} > 0$, $\alpha_{i1} + \alpha_{i2} = 1$ for $i = 1, 2, \ldots, K$. The α_{ij} can be chosen for equal tails, for shortest, or for shortest unbiased confidence intervals when $K - 1$ of the θ_i are zero. The confidence interval in (2.2) is exact when (1) any $K - 1$ of the θ_i are zero; (2) when any $K - 1$ of the $n_i \to \infty$. When any M of the $\theta_i = 0$ for $M < K$ the resulting confidence interval reduces to the MLS confidence interval for a nonnegative linear combination of the remaining $K - M$ variances θ_i; also when any M of the $n_i \to \infty$ for $M < K$ the resulting

confidence interval reduces to the MLS confidence interval for a non-negative linear combination of the remaining K - M variances ψ_i.

It is seen that the probability coverages associated with the Satterthwaite, Welch, and MLS approximate confidence intervals for $c\theta_1 + \theta_2$ depend on c and the population parameters θ_1, θ_2 only through the unknown parameter ρ defined by $\rho = c\theta_1/(c\theta_1 + \theta_2)$. Clearly $0 \leq \rho \leq 1$. Simpson's rule with interval size h = 0.01 was used to evaluate the integral (the probability) given in Fleiss (1971) for the different functions z(w) given by the Satterthwaite, Welch, and MLS methods. The IMSL subroutine MDCH was used to compute the chi-square distribution. The values of ρ used were ρ = 0.0 (0.1) 1.0; all combinations of the following values of n_1 and n_2 were examined for $1 - \alpha$ equal to .90 and .95.

n_1: 4, 5, 6, 7, 8, 9, 10, 15, 20, 30

n_2: 4, 5, 6, 7, 8, 9, 10, 15, 20, 30

The results are given in Tables 2 and 3. The entries are the ranges that the confidence coefficients vary as the unknown parameter ρ varies in the set 0.0 (.1) 1.0. The column headed MLS1 is for "equal tails" confidence intervals; the column headed MLS2 is for "shortest unbiased" confidence intervals; the column headed MLS3 is for "shortest" confidence intervals.

To evaluate the expected lengths a simulation study was conducted. One thousand chi-square random numbers were generated using the IMSL subroutine GGCSS (chi-square random deviate generator) for each pair of values on n_1 and n_2 listed below

n_1 4, 4, 4, 8, 8, 16, 16

n_2 4, 8, 30, 8, 30, 32, 60

From these random numbers the three ratios, r_1, r_2 and r_3 were evaluated for $1 - \alpha = .90$ and $1 - \alpha = .95$ where

$$\frac{\text{Average length of MLSi confidence interval}}{\text{Average length of Welch confidence interval}} = r_i$$

The results are recorded in Tables 4 and 5 broken down for $\rho = 0.0(.1)1.0$. The ratios depend on c, θ_1 and θ_2 only through the parameter ρ.

Some conclusions from the formulas are as follows:

(1) The results are for all values of $c \geq 0$.

(2) Only the MLS methods give correct asymptotic results for large n_i and small n_j for $i \neq j$.

(3) When $\rho = 0$ only the Satterthwaite and MLS methods are exact.

(4) When $\rho = 1$ only the Satterthwaite and MLS methods are exact.

(5) The MLS methods are easy to compute even for nonnegative linear combinations of K variances.

Some conclusions from the tables are as follows:

(1) The confidence coefficients for the Welch method are closer to the nominal values than the Satterthwaite method but the Welch method is more difficult to compute.

(2) The confidence coefficients for the Welch and Satterthwaite methods can fall several points <u>below</u> the nominal level. This is undesirable.

(3) The confidence coefficients for the MLS methods appear to be greater than or equal to the nominal values. This is a desirable characteristic if the average width is satisfactory.

(4) The MLS2 and MLS3 methods give confidence intervals whose average widths are generally smaller (and sometimes significantly smaller) than the average widths of the Welch method.

3. An Iteration Method

Trickett and Welch [1954] derive a method of calculating critical values for the problem of setting confidence intervals on the difference of two mean values of normal populations with unknown and unequal variances. A similar procedure can be used to find confidence intervals on $c\theta_1 + \theta_2$. Suppose there exist functions $z_1(r)$, $z_2(r)$ where $r = cS_1^2/(cS_1^2 + S_2^2)$ such that an exact solution exists for equation (3.1).

$$
\begin{aligned}
1 - \alpha &= P[z_1(r) \leq (cS_1^2 + S_2^2)/(c\theta_1 + \theta_2) \leq z_2(r)] \\
&= P[(cS_1^2 + S_2^2)/(c\theta_1 + \theta_2) \leq z_2(r)] - \\
&\quad P[cS_1^2 + S_2^2)/(c\theta_1 + \theta_2) \leq z_1(r)]
\end{aligned}
\tag{3.1}
$$

Suppose that a first approximation to z_2 is given by, say z_0 (known); we write

$$z_2(r) = z_0(r) + z(r)$$

Then, by using a Taylor expansion and ignoring the powers of $z(r)$ higher than the first, Equation (3.1) becomes

$$
\begin{aligned}
1 - \alpha &= \int_0^1 H_{n_1 + n_2}[z_0(r)/(\rho(1-w)/n_1 + (1-\rho)w/n_2]p(w)dw \\
&\quad + \int_0^1 [z(r)/(\rho(1-w)/n_1 + (1-\rho)w/n_2]H'_{n_1 + n_2}[z_0(r) \\
&\qquad /\rho(1-w)/n_1 + (1-\rho)w/n_2)]p(w)dw \\
&\quad - \int_0^1 H_{n_1 + n_2}[z_1(r)/(\rho(1-w)/n_1 + (1-\rho)w/n_2)]p(w)dw
\end{aligned}
\tag{3.2}
$$

If one can obtain $z(r)$ to satisfy Equation (3.2) then an improved approximation $z_0(r) + z(r)$ to the solution might be expected. In order to solve for $z(r)$, a simplification to the second integral in Equation (3.2) is made as follows.

The distribution $p(w)$ has mean value at $w = n_2/(n_1 + n_2)$. It follows that for large n_1 and n_2 the distribution will be closely concentrated about $n_2/(n_1 + n_2)$ and therefore an integral of the form $\int_0^1 g(w)p(w)dw$ will be approximately equal to $g(n_2/(n_1 + n_2))$.

Now when $w = n_2/(n_1 + n_2)$ then $r = cS_1^2/(cS_1^2 + S_2^2) = \rho$, and the above simplification applied to the second integral of Equation (3.2) gives $(n_1 + n_2)z(\rho)H'_{n_1 + n_2}((n_1 + n_2)\, z_0(\rho))$. If I_1 and I_2 denote the first and third integrals respectively in Equation (3.2) then

$$1 - \alpha = (I_1 - I_2) + (n_1 + n_2)z(\rho)H'_{n_1 + n_2}((n_1 + n_2)z_0(\rho))$$

i.e.

$$z(\rho) = [1 - \alpha - (I_1 - I_2)]/[(n_1 + n_2)H'_{n_1 + n_2}((n_1 + n_2)z_0(\rho))] \qquad (3.3)$$

where the quantity $H'_{n_1 + n_2}((n_1 + n_2)z_0(\rho))$ is the ordinate of the chi-square distribution with $n_1 + n_2$ degrees of freedom evaluated at $(n_1 + n_2)z_0(\rho)$. By taking a sufficient number of values of ρ, say $\rho = 0\ (0.1)\ 1.0$, the table of $z(\rho)$ can be constructed. Since $z(r)$ is the same function of r as $z(\rho)$ is of ρ this is equivalent to tabulating $z(r)$ for $r = 0\ (0.1)\ 1.0$.

The function $z(r)$ thus obtained will not exactly satisfy Equation (3.2) because of the crude approximation made to the second integral, but may bring us appreciably closer to a solution of Equation (3.1). With this fresh approximation one can start a further cycle of the process and continue.

By letting $z_0(r) = 1/\{1 - [r^2(1 - 1/F_{\alpha/2:\, n_1,\, \infty})^2 + (1 - r)^2 (1 - 1/F_{\alpha/2:\, n_2,\, \infty})^2]^{\frac{1}{2}}\}$ and $z_1(r) = 1/\{1 + [r^2(1/F_{1-\alpha/2:\, n_1,\, \infty} - 1)^2 + (1 - r)^2(1/F_{1-\alpha/2:\, n_2,\, \infty} - 1)^2]^{\frac{1}{2}}\}$, i.e. the confidence limits of the

MLS1 method, one can evaluate I_1 and I_2 for fixed ρ and perform the iteration. For some selected n_1 and n_2 and for $1 - \alpha = 0.90$ and 0.95 tables of $z(r)$ were obtained by using Simpson's rule to evaluate the integrals I_1 and I_2, and Lagrange interpolation formula to calculate $z(r)$ for sample values of r. The iteration terminates when the actual confidence coefficients are accurate to five decimal places for all ρ, i.e. for $1 - \alpha = 0.90$ the resulting confidence coefficients are within the range of 0.89995 and 0.90004 for all ρ and for each n_1 and n_2. Tables 6 and 7 contain the values of $z(r)$ for $1 - \alpha = 0.90$ and 0.95 respectively. Thus, with the aid of the tables of $z(r)$, one can conclude that the confidence interval

$$\begin{aligned} &1/\{1 + [r^2(1/F_{1-\alpha/2:\, n_1,\, \infty} - 1)^2 + (1-r)^2(1/F_{1-\alpha/2:\, n_2,\, \infty} - 1)^2]^{\frac{1}{2}}\} \\ &\le (cS_1^2 + S_2^2)/(c\theta_1 + \theta_2) \le 1/\{1 - [r^2(1 - 1/F_{\alpha/2:\, n_1,\, \infty})^2 \\ &+ (1-r)^2(1 - 1/F_{\alpha/2:\, n_2,\, \infty})^2]^{\frac{1}{2}}\} + z(r) \end{aligned} \tag{3.4}$$

has the desirable property that its probability of coverage is essentially equal to $1 - \alpha$. From this one can obtain an "exact" confidence interval on $c\theta_1 + \theta_2$ where $c \geq 0$.

Table 1

Ranges of Confidence Coefficients for Welch Method

$1 - \alpha = .95$

n_1	n_2	Confidence Coefficients
4	100	.9054 - .9558
8	100	.9369 - .9523

Table 2

Ranges of Confidence Coefficients (Times 10^4) for Satterthwaite, Welch and MLS Procedures

$1 - \alpha = 0.90$

n_1	n_2	S	W	MLS1	MLS2	MLS3
4	4	8855-9244	8892-9071	9000-9208	9000-9288	9000-9463
	5	8786-9226	8888-9069	9000-9207	9000-9269	9000-9439
	6	8739-9210	8859-9069	9000-9205	9000-9254	9000-9420
	7	8707-9202	8843-9068	9000-9202	9000-9248	9000-9404
	8	8684-9194	8838-9065	9000-9200	9000-9243	9000-9390
	9	8667-9186	8841-9063	9000-9198	9000-9238	9000-9379
	10	8654-9179	8821-9086	9000-9195	9000-9233	9000-9370
	15	8542-9157	8769-9157	9000-9186	9000-9215	9000-9337
	20	8479-9146	8738-9203	9000-9194	9000-9201	9000-9309
	30	8403-9128	8691-9186	9000-9188	9000-9190	9000-9291
5	5	8860-9212	8928-9071	9000-9174	9000-9246	9000-9412
	6	8821-9198	8907-9067	9000-9174	9000-9233	9000-9390
	7	8793-9185	8892-9064	9000-9174	9000-9233	9000-9374
	8	8773-9178	8886-9064	9000-9174	9000-9214	9000-9361
	9	8758-9172	8887-9062	9000-9174	9000-9206	9000-9349
	10	8746-9167	8889-9059	9000-9173	9000-9203	9000-9339
	15	8667-9141	8844-9106	9000-9167	9000-9191	9000-9303
	20	8610-9134	8830-9128	9000-9160	9000-9181	9000-9285
	30	8554-9117	8789-9154	9000-9160	9000-9165	9000-9263
6	6	8873-9187	8936-9066	9000-9152	9000-9213	9000-9369
	7	8848-9177	8921-9062	9000-9148	9000-9205	9000-9352
	8	8830-9167	8913-9057	9000-9148	9000-9197	9000-9337
	9	8831-9170	8911-9059	9000-9149	9000-9190	9000-9326
	10	8805-9155	8915-9056	9000-9149	9000-9186	9000-9316
	15	8749-9134	8887-9069	9000-9149	9000-9169	9000-9281
	20	8696-9121	8885-9086	9000-9146	9000-9162	9000-9260
	30	8644-9110	8847-9095	9000-9136	9000-9150	9000-9239

Table 2 (Continued)

$1 - \alpha = 0.90$

n_1	n_2	S	W	MLS1	MLS2	MLS3
7	7	8885-9168	8941-9060	9000-9137	9000-9188	9000-9334
	8	8869-9160	8933-9056	9000-9134	9000-9182	9000-9320
	9	8856-9151	8929-9052	9000-9131	9000-9176	9000-9307
	10	8846-9144	8930-9050	9000-9131	9000-9171	9000-9296
	15	8807-9127	8915-9046	9000-9131	9000-9152	9000-9263
	20	8757-9111	8910-9059	9000-9131	9000-9146	9000-9242
	30	8707-9102	8885-9067	9000-9126	9000-9137	9000-9218
8	8	8897-9153	8947-9054	9000-9124	9000-9167	9000-9305
	9	8885-9146	8942-9051	9000-9122	9000-9163	9000-9293
	10	8875-9139	8941-9047	9000-9120	9000-9159	9000-9282
	15	8847-9121	8935-9041	9000-9119	9000-9143	9000-9247
	20	8802-9107	8927-9042	9000-9118	9000-9132	9000-9228
	30	8754-9096	8911-9048	9000-9117	9000-9126	9000-9178
9	9	8906-9140	8953-9049	9000-9113	9000-9152	9000-9281
	10	8897-9134	8950-9046	9000-9111	9000-9148	9000-9270
	15	8871-9114	8949-9038	9000-9106	9000-9134	9000-9234
	20	8837-9103	8940-9032	9000-9106	9000-9123	9000-9215
	30	8791-9089	8930-9036	9000-9105	9000-9116	9000-9192
10	10	8915-9129	8958-9044	9000-9104	9000-9140	9000-9260
	15	8889-9108	8960-9035	9000-9098	9000-9126	9000-9223
	20	8864-9099	8950-9031	9000-9098	9000-9117	9000-9204
	30	8820-9083	8944-9028	9000-9084	9000-9107	9000-9182
15	15	8942-9094	8977-9028	9000-9072	9000-9100	9000-9189
	20	8931-9081	8979-9021	9000-9072	9000-9090	9000-9168
	30	8905-9069	8975-9019	9000-9067	9000-9083	9000-9145
20	20	8956-9074	8986-9019	9000-9055	9000-9078	9000-9148
	30	8946-9060	8984-9014	9000-9054	9000-9068	9000-9124
30	30	8971-9050	8994-9011	9000-9038	9000-9054	9000-9103

Table 3

Ranges of Confidence Coefficients (Times 10^4) for Satterthwaite, Welch and MLS Procedures

$1 - \alpha = 0.95$

n_1	n_2	S	W	MLS1	MLS2	MLS3
4	4	9393-9651	9379-9507	9491-9614	9500-9685	9500-9800
	5	9326-9641	9382-9515	9492-9609	9500-9674	9500-9790
	6	9275-9633	9356-9522	9495-9612	9500-9666	9500-9781
	7	9238-9626	9336-9525	9497-9615	9500-9658	9500-9772
	8	9212-9622	9323-9526	9498-9617	9500-9652	9500-9764
	9	9192-9620	9318-9526	9497-9617	9500-9646	9500-9758
	10	9176-9616	9318-9528	9497-9617	9500-9641	9500-9754
	15	9081-9602	9316-9533	9497-9617	9500-9638	9500-9735
	20	9008-9597	9247-9530	9498-9618	9500-9636	9500-9720
	30	8933-9583	9191-9541	9500-9619	9500-9631	9500-9699
5	5	9388-9632	9419-9524	9492-9602	9500-9656	9500-9770
	6	9348-9625	9408-9527	9494-9599	9500-9650	9500-9763
	7	9319-9618	9392-9528	9495-9596	9500-9645	9500-9755
	8	9297-9613	9382-9530	9497-9594	9500-9640	9500-9748
	9	9280-9610	9378-9530	9499-9591	9500-9636	9500-9742
	10	9267-9607	9377-9530	9499-9591	9500-9632	9500-9736
	15	9201-9593	9339-9583	9498-9592	9500-9615	9500-9713
	20	9136-9588	9329-9617	9498-9597	9500-9613	9500-9701
	30	9073-9579	9283-9640	9500-9597	9500-9587	9500-9683
6	6	9394-9618	9439-9530	9494-9590	9500-9632	9500-9749
	7	9369-9612	9424-9530	9495-9589	9500-9629	9500-9740
	8	9350-9607	9415-9529	9496-9587	9500-9626	9500-9733
	9	9335-9602	9410-9530	9497-9586	9500-9624	9500-9727
	10	9323-9600	9409-9530	9499-9584	9500-9621	9500-9721
	15	9279-9588	9386-9565	9499-9581	9500-9608	9500-9700
	20	9221-9579	9375-9587	9499-9587	9500-9598	9500-9685
	30	9163-9574	9342-9609	9499-9592	9500-9583	9500-9668

Table 3 (Continued)

$1 - \alpha = 0.95$

n_1	n_2	S	W	MLS1	MLS2	MLS3
7	7	9402-9607	9445-9531	9495-9580	9500-9614	9500-9730
	8	9386-9602	9436-9530	9496-9579	9500-9613	9500-9722
	9	9372-9597	9431-9528	9497-9579	9500-9611	9500-9715
	10	9362-9593	9429-9528	9498-9578	9500-9609	9500-9709
	15	9331-9583	9416-9526	9500-9573	9500-9601	9500-9688
	20	9279-9574	9404-9524	9499-9571	9500-9593	9500-9674
	30	9225-9569	9382-9523	9498-9575	9500-9581	9500-9659
8	8	9411-9597	9451-9530	9496-9571	9500-9602	9500-9714
	9	9399-9593	9445-9529	9496-9571	9500-9599	9500-9707
	10	9389-9590	9443-9527	9497-9571	9500-9598	9500-9701
	15	9361-9578	9436-9525	9500-9569	9500-9593	9500-9679
	20	9322-9571	9424-9521	9500-9565	9500-9587	9500-9665
	30	9272-9564	9409-9521	9499-9566	9500-9575	9500-9650
9	9	9419-9590	9456-9528	9497-9563	9500-9593	9500-9699
	10	9410-9586	9453-9527	9497-9563	9500-9589	9500-9693
	15	9383-9574	9451-9524	9500-9564	9500-9585	9500-9671
	20	9370-9568	9439-9520	9500-9559	9500-9582	9500-9658
	30	9307-9559	9429-9520	9499-9556	9500-9573	9500-9642
10	10	9426-9583	9461-9526	9497-9557	9500-9586	9500-9687
	15	9400-9570	9462-9522	9500-9559	9500-9578	9500-9665
	20	9380-9565	9450-9520	9500-9558	9500-9576	9500-9651
	30	9335-9555	9444-9517	9500-9554	9500-9570	9500-9636
15	15	9449-9561	9478-9518	9500-9537	9500-9561	9500-9642
	20	9438-9554	9479-9514	9500-9537	9500-9555	9500-9628
	30	9415-9547	9474-9513	9500-9537	9500-9552	9500-9612
20	20	9462-9549	9487-9513	9500-9527	9500-9547	9500-9614
	30	9452-9540	9484-9510	9500-9529	9500-9541	9500-9598
30	30	9475-9534	9494-9508	9500-9519	9500-9532	9500-9582

Table 4

Ratios of Expected Length of MSL to Welch Confidence Intervals on $c\theta_1 + \theta_2$

$1 - \alpha = 0.90$

ρ	$n_1=4,\ n_2=4$			$n_1=4,\ n_2=8$			$n_1=4,\ n_2=30$		
	MLS1	MLS2	MLS3	MLS1	MLS2	MLS3	MLS1	MLS2	MLS3
0.0	1.08	.86	.74	1.01	.91	.83	1.00	.97	.94
0.2	1.03	.83	.71	1.17	1.03	.92	1.20	1.03	.94
0.4	1.18	.94	.81	1.17	.98	.86	.85	.69	.60
0.6	1.19	.95	.81	.95	.77	.68	.76	.61	.53
0.8	1.04	.83	.71	.90	.73	.62	.88	.70	.60
1.0	1.08	.86	.74	1.08	.86	.74	1.08	.86	.74

ρ	$n_1=8,\ n_2=8$			$n_1=8,\ n_2=30$			$n_1=16,\ n_2=32$		
	MLS1	MLS2	MLS3	MLS1	MLS2	MLS3	MLS1	MLS2	MLS3
0.0	1.01	.91	.83	1.00	.97	.95	1.00	.98	.95
0.2	1.03	.92	.84	1.12	1.06	1.01	1.04	1.01	.98
0.4	1.15	1.04	.94	1.07	.98	.91	1.08	1.04	1.00
0.6	1.16	1.04	.95	.98	.89	.81	1.03	.98	.93
0.8	1.03	.93	.84	.98	.88	.80	1.00	.95	.90
1.0	1.01	.91	.83	1.01	.91	.83	1.00	.95	.90

Table 5

Ratios of Expected Length of MLS to Welch Confidence Intervals on $c\theta_1 + \theta_2$

$1 - \alpha = 0.95$

ρ	n_1=4, n_2=4			n_1=4, n_2=8			n_1=4, n_2=30		
	MLS1	MLS2	MLS3	MLS1	MLS2	MLS3	MLS1	MLS2	MLS3
0.0	1.24	.98	.86	1.02	.92	.84	1.00	.98	.95
0.2	1.00	.80	.69	1.20	1.04	.94	1.09	.92	.83
0.4	1.11	.88	.77	1.04	.86	.76	.48	.39	.34
0.6	1.12	.89	.78	.72	.58	.51	.45	.36	.32
0.8	1.01	.80	.70	.78	.62	.54	.76	.60	.53
1.0	1.24	.98	.86	1.24	.98	.86	1.24	.98	.86

ρ	n_1=8, n_2=8			n_1=8, n_2=30			n_1=16, n_2=32		
	MLS1	MLS2	MLS3	MLS1	MLS2	MLS3	MLS1	MLS2	MLS3
0.0	1.02	.92	.84	1.00	.98	.95	1.00	.98	.95
0.2	1.02	.92	.84	1.15	1.09	1.03	1.05	1.02	.98
0.4	1.18	1.06	.97	1.06	.97	.90	1.10	1.06	1.01
0.6	1.19	1.07	.97	.94	.85	.78	1.03	.98	.93
0.8	1.03	.92	.84	.96	.87	.79	.99	.94	.90
1.0	1.02	.92	.84	1.02	.92	.84	1.00	.95	.91

Table 6

Values of $z(r)$ for $1 - \alpha = 0.90$

r		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
n_1	n_2											
4	4	0.0	-0.19049	-0.15634	-0.07489	-0.03485	-0.02313	-0.03485	-0.07489	-0.15634	-0.19049	0.0
4	6	0.0	-0.14585	-0.11823	-0.05873	-0.02951	-0.02419	-0.04000	-0.07855	-0.12829	-0.10603	0.0
4	8	0.0	-0.12071	-0.09783	-0.04906	-0.02526	-0.02324	-0.04189	-0.07892	-0.10007	-0.05647	0.0
4	10	0.0	-0.10581	-0.08382	-0.04272	-0.02381	-0.02423	-0.04310	-0.07474	-0.08271	-0.04275	0.0
5	5	0.0	-0.12660	-0.12341	-0.06600	-0.03349	-0.02357	-0.03349	-0.06600	-0.12341	-0.12660	0.0
5	7	0.0	-0.10302	-0.09920	-0.05287	-0.02756	-0.02243	-0.03602	-0.06756	-0.09751	-0.07050	0.0
5	9	0.0	-0.08893	-0.08428	-0.04479	-0.02402	-0.02137	-0.03597	-0.06411	-0.07581	-0.03962	0.0
6	6	0.0	-0.08524	-0.09815	-0.05890	-0.03129	-0.02230	-0.03129	-0.05890	-0.09815	-0.08624	0.0
6	8	0.0	-0.07403	-0.08218	-0.04800	-0.02599	-0.02067	-0.03194	-0.05783	-0.07692	-0.04934	0.0
6	10	0.0	-0.06627	-0.07137	-0.04104	-0.02287	-0.01975	-0.03131	-0.05310	-0.05980	-0.02059	0.0
7	7	0.0	-0.06085	-0.07932	-0.05243	-0.02871	-0.02052	-0.02871	-0.05243	-0.07932	-0.06085	0.0
7	9	0.0	-0.05454	-0.06832	-0.04354	-0.02434	-0.01907	-0.02844	-0.04966	-0.06219	-0.03632	0.0
8	8	0.0	-0.04460	-0.06511	-0.04647	-0.02620	-0.01888	-0.02620	-0.04647	-0.06511	-0.0446	0.0
8	10	0.0	-0.04130	-0.05731	-0.03936	-0.02264	-0.01766	-0.02554	-0.04296	-0.05130	-0.02788	0.0
10	10	0.0	-0.02657	-0.04573	-0.03652	-0.02192	-0.01625	-0.02192	-0.03652	-0.04573	-0.02657	0.0

Table 7

Values of $z(r)$ for $1 - \alpha = 0.95$

n_1	n_2 \ r	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
4	4	0.0	-0.21091	-0.08775	0.00704	0.04811	0.05956	0.04811	0.00704	-0.08775	-0.21091	0.0
4	6	0.0	-0.16346	-0.06556	0.00261	0.03510	0.04114	0.02758	-0.00392	-0.09332	-0.15892	0.0
4	8	0.0	-0.13426	-0.05625	0.00126	0.02926	0.03532	0.02053	-0.02149	-0.09275	-0.10331	0.0
4	10	0.0	-0.12861	-0.04765	0.00096	0.02420	0.02712	0.01302	-0.01020	-0.09120	-0.10012	0.0
5	5	0.0	-0.16076	-0.08147	-0.00017	0.03179	0.04078	0.03179	-0.00017	-0.08147	-0.16076	0.0
5	7	0.0	-0.12811	-0.06613	-0.00303	0.02548	0.03212	0.02002	-0.01295	-0.08087	-0.11190	0.0
5	9	0.0	-0.10703	-0.05772	-0.00415	0.02209	0.02758	0.01336	-0.02430	-0.07657	-0.07788	0.0
6	6	0.0	-0.12048	-0.07375	-0.00723	0.02235	0.03128	0.02235	-0.00723	-0.07375	-0.12408	0.0
6	8	0.0	-0.09983	-0.06175	-0.00725	0.01860	0.02573	0.01456	-0.01723	-0.06925	-0.08200	0.0
6	10	0.0	-0.08530	-0.05494	-0.00746	0.01687	0.02194	0.00888	-0.02328	-0.06049	-0.05420	0.0
7	7	0.0	-0.09107	-0.06536	-0.01185	0.01613	0.02516	0.01613	-0.01185	-0.06536	-0.09107	0.0
7	9	0.0	-0.07809	-0.05605	-0.01012	0.01371	0.02092	0.01089	-0.01841	-0.05911	-0.06146	0.0
8	8	0.0	-0.06991	-0.05760	-0.01424	0.01196	0.02064	0.01196	-0.01424	-0.05760	-0.06991	0.0
8	10	0.0	-0.06180	-0.05018	-0.01186	0.01014	0.01712	0.00803	-0.01864	-0.05155	-0.04872	0.0
10	10	0.0	-0.04365	-0.04462	-0.01530	0.00674	0.01415	0.00674	-0.01530	-0.04462	-0.04365	0.0

REFERENCES

Burdick, R.K. and Sielken, R.L. Jr. (1978). "Exact Confidence intervals for linear combinations of variance components in nested classifications," Journal of the American Statistical Association, 73,632-635.

Fleiss, Joseph L. (1971), "Distribution of a Linear Combination of Independent Chi Squares," Journal of the American Statistical Association, 66, 142-144.

Graybill, Franklin A. (1976), Theory and Application of the Linear Model, North Scituate, Mass: Duxbury Press.

Huitson, A. (1955), "A method of Assigning Confidence Limits to Linear Combination of Variances," Biometrika, 42, 471-479.

John, S. (1973), "Critical Values for Inferences about Normal Dispersion," Australian Journal of Statistics, 15, 71-79.

Satterthwaite, Franklin E. (1941), "Synthesis of Variance," Psychometrika, 6, 309-316.

Satterthwaite, Franklin E. (1946), "An Approximate Distribution of Estimates of Variance Components," Biometrics Bulletin, 2, 110-114.

Smith, H. Fairfield (1936), "The Problem of Comparing the Results of two Experiments with Unequal Errors," Journal of the Council of Scientific and Industrial Research, 9, 211-212.

Tate, R. F. and Klett, G. W. (1959), "Optimal Confidence Intervals for the Variance of a Normal Distribution," Journal of the American Statistical Association, 54, 674-682.

Welch, B. L. (1956), "On Linear Combinations of Several Variances," Journal of the American Statistical Association, 51, 132-148.